Splitting Methods

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Numerical Analysis

Within Numerical Analysis, methods for finding numerical approximations to mathematical problems are constructed and analyzed. The subject can be seen as a discipline within applied mathematics as well as within computer science. Numerical methods are used in both science and engineering, e.g. for simulation, validation and dimensioning.

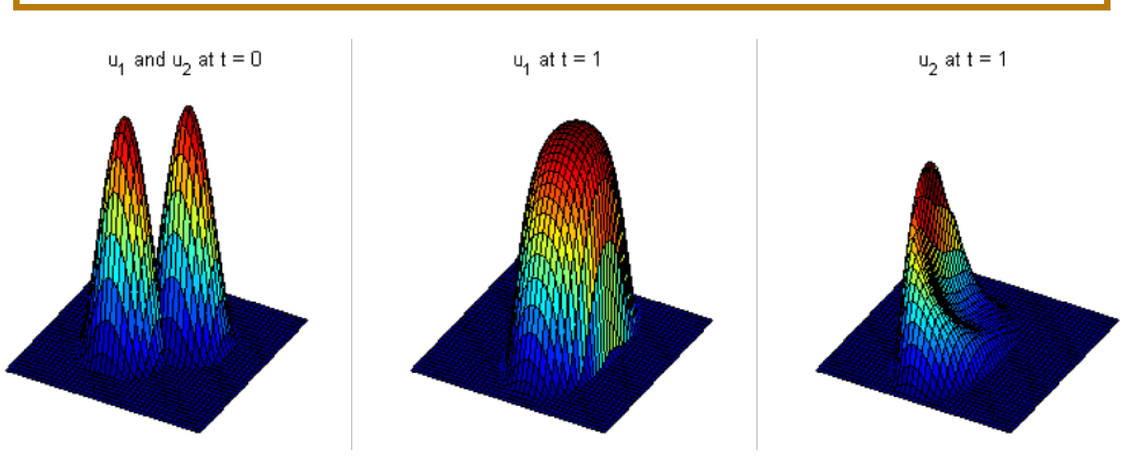
Why Splitting Methods?

Partial differential equations (PDEs) constitute one, if not the, main building block when modeling physical processes. Examples of PDEs are the heat equations, the wave equation and the diffusion-reaction equation

$$\dot{u} = \Delta u + f(u).$$

Applications in engineering and medicine, where several physical processes interact, often result in models consisting of large systems of equations. For example, a standard air pollution model consists of a system with approximately one hundred coupled 3D advection-diffusion-reaction equations (one for every trace gas concentration). Due to the sheer size of the model, the usage of direct, well-known approximation methods like the Finite Element Method is not feasible and one needs to partition or split the problem.

What Are Splitting Methods?



Lets illustrate splitting with an example. Consider a simple diffusion-reaction system modeling two competing insect species (see the equation above). The graphs depict the initial population densities (height) and the populations after one time unit, respectively. As seen from the simulations, the first insect species out-competes the second one. The solution is received by first neglecting the diffusion terms, i.e., the species interact regardless of the surrounding population density, and thereafter letting the two species spread out in the habitat without any interaction. This procedure is repeated for short time intervals and is a splitting procedure.

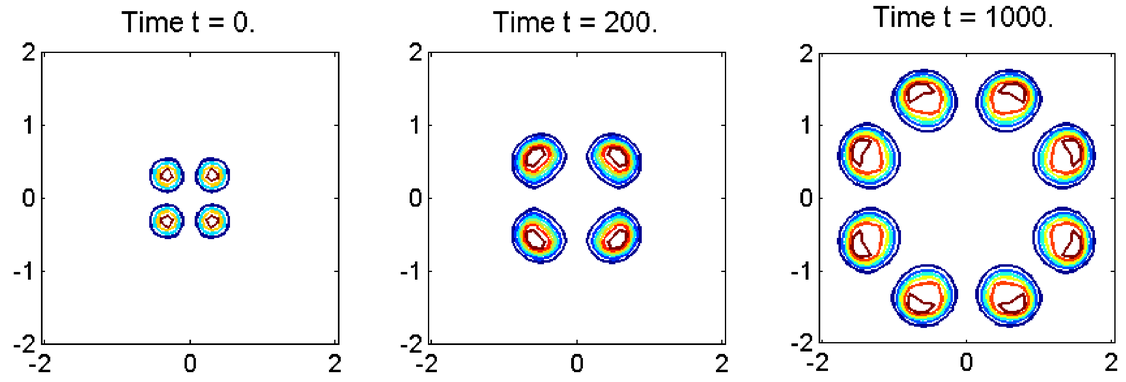
It is of utmost importance to note that by doing this splitting procedure we introduce an error. The analysis of this error is central in the research of splitting methods.

Ex: Gray-Scott Model

The Gray-Scott model describes two chemicals in interaction. Let u and v be the respective concentrations of these chemicals. The model is then

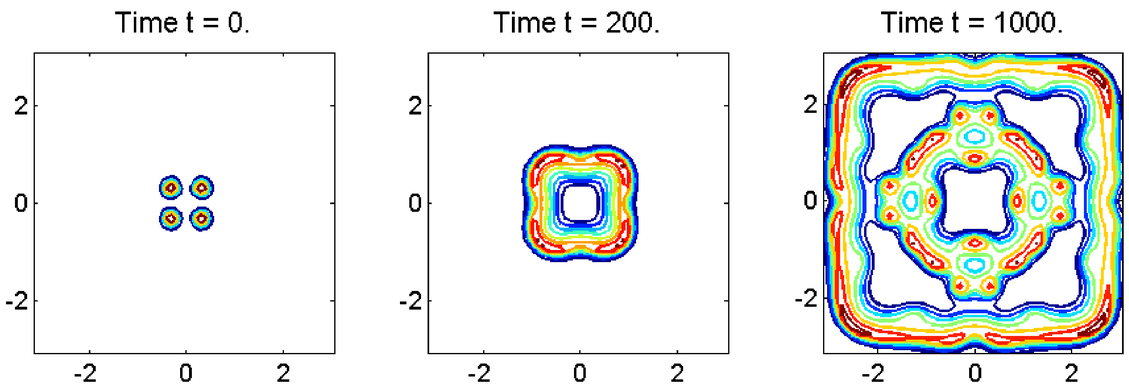
$$\dot{u} = D_1 \Delta u - uv^2 + \gamma(1-u)$$
$$\dot{v} = D_2 \Delta v + uv^2 - (\gamma + \kappa)v$$

where D_1, D_2, γ and κ are parameters. A solution for parameters $D_1 = 8 \cdot 10^{-4}$, $D_2 = 4 \cdot 10^{-4}$, $\gamma = 0.024$ and $\kappa = 0.06$ can be seen below in contour plots of the concentration v. Starting with an initial concentration consisting of four peaks the chemicals spreads and after 1000 s there are eight peaks.



A stable and accurate numerical method may be slow when solving the above equation due to the non-linear terms. Using splitting we can first neglect the reaction and only consider the diffusion parts. After the chemicals have diffused we can let them interact, i.e., we consider the last two terms in the equation. This second part is now as simple as solving a system of two second order equations.

The solution of the Gray-Scott equations is very sensible to parameter values. Changing the value of κ (only) to 0.053 gives a totally different solution. The concentration v can be seen in the contour plots below.



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