Splitting methods

Tony Stillfjord

Lund University
Centre for Mathematical Sciences
Numerical Analysis

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Contents

Numerical Analysis

Our group

My work

Numerical Analysis

"Numerical analysis aims to construct and analyze quantitative methods for the automatic computation of approximate solutions to mathematical problems."

— Gustaf Söderlind

"Numerical analysis is the area of mathematics and computer science that creates, analyzes, and implements algorithms for solving numerically the problems of continuous mathematics."

— Kendall E. Atkinson

"Numerical analysis is the study of algorithms for the problems of continuous mathematics."

— Lloyd N. Trefethen

Numerical Analysis topics

Numerical linear algebra

- Solving systems of equations
- Computing eigenvalues
- Matrix factorizations, functions of matrices

Approximation theory

- Interpolation, extrapolation
- Numerical integration

Optimization

- Min/max of real-valued functions
- Possibly with constraints

Differential equations

- ODEs, PDEs
- Integral equations
- DAEs, SDEs, DDEs

Research areas at Numerical Analysis in Lund

Main focus: Differential equations

We work with

- Adaptivity (Gustaf Söderlind)
- DAEs (Claus Führer)
- Multistep methods (Carmen Arévalo)
- Integral equations (Johan Helsing)
- Real-time simulation (Christian Andersson)
- Splitting methods (Eskil Hansen, Erik Henningsson, Tony Stillfjord)

and more

My work: Laplacian example

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t,x) = \Delta u(t,x), \quad x \in [0,1], \quad t \in [0,1]$$
$$u(t,0) = u(t,1) = 0$$
$$u(0,x) = f(x)$$

Easy and fast to solve by Fast Fourier Transform (FFT) techniques

But what about this?

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,x) = \Delta u(t,x) + g(u), \quad x \in [0,1], \quad t \in [0,1]$$
$$u(t,0) = u(t,1) = 0$$
$$u(0,x) = f(x)$$

$g(u)$ non-linear, but "nice", non-stiff

FFT-techniques do not work (or complicated and specific)

A typical problem

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + Bu(t), \quad t \in [0,1],$$
$$u(0) = \eta$$

Abstract evolution equation

Space dependency and boundary conditions hidden in the operators A and B

Full problem difficult/expensive

Sub-problems $\quad \frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t)$, $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Bu(t)$ $\quad$ easy/cheap

Splitting methods: Lie splitting

Iterate between the subproblems

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = Av(t), \quad t \in [0, \Delta t], \qquad v(0) = \eta$$

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = Bw(t), \quad t \in [0, \Delta t], \qquad w(0) = v_1$$

$$\rightarrow v_1 \approx v(\Delta t) \qquad \rightarrow w_1 \approx w(\Delta t)$$

$$\dot{u} = Au + Bu$$
$$u_1 = w_1 \approx u(\Delta t)$$

Splitting methods: Lie splitting

Iterate between the subproblems

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = Av(t), \quad t \in [0, \Delta t], \qquad \frac{\mathrm{d}}{\mathrm{d}t} w(t) = Bw(t), \quad t \in [0, \Delta t],$$

$$v(0) = w_1 \qquad\qquad w(0) = v_2$$

$$\to v_2 \approx v(\Delta t) \qquad\qquad \to w_2 \approx w(\Delta t)$$

$$\dot{u} = Au + Bu$$
$$u_2 = w_2 \approx u(2\Delta t)$$

Splitting methods: Lie splitting

Iterate between the subproblems

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = Av(t), \quad t \in [0, \Delta t], \qquad v(0) = w_{n-1}$$

$$\rightarrow v_n \approx v(\Delta t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = Bw(t), \quad t \in [0, \Delta t], \qquad w(0) = v_n$$

$$\rightarrow w_n \approx w(\Delta t)$$

$$\dot{u} = Au + Bu$$
$$u_n = w_n \approx u(n\Delta t) = u(T)$$

Splitting methods: convergence

For bounded operators A and B (think fixed spatial discretization), Use Taylor expansion to prove convergence with order

$$\|u_n - u(n\Delta t)\| \leq C(\Delta t)^p$$

But $C \to \infty$ as discretization becomes finer !

Taylor expansion does not work for unbounded operators

Until recently: Only order theory for classical splitting methods (i.e. for bounded operators)

In our group (Eskil Hansen)

Under certain conditions (linear maximal dissipative operators, etc.):

Order is preserved for classical splitting methods:

$$\|u_n - u(n\Delta t)\| \leq C(\Delta t)^p$$

C independent of spatial discretization mesh width!

IMEX Euler

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + Bu(t), \quad t \in [0, 1],$$
$$u(0) = \eta$$

A (non-linear) unbounded dissipative operator, like $\Delta\big(|u|^m u\big)$
B Lipschitz continuous operator

Solve $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t)$ by Implicit Euler
Solve $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Bu(t)$ by Explicit Euler

$\rightarrow$ Same order as Implicit Euler for full problem (≤ 1)

(Eskil Hansen and Tony Stillfjord 2012)

Delay Differential Equations

Now trying to prove similar results for splitting DDEs, for example

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + u(t-1), \quad t \in [0,1],$$
$$u(0) = \eta$$
$$u(\tau) = f(\tau), \quad \tau \in [-1,0]$$

Thank you